Section 3.2 Applications of Exponential Functions

Exploration 3.7

After searching for work over the first two months of summer vacation, you finally receive two offers to start work on August 1. The first potential employer offers to pay you \$100 on the first day and then give you a raise of \$500 per day for each subsequent day that you work. The second potential employer offers to pay you 5¢on the first day and to double your salary each subsequent day that you work. If you were to work every day in August, what would your salary be on August 8 and on August 31 at each job? What would the graph of salary as a function of days since August 1 look like in each case?

Quantities that change by having the same amount added to (or subtracted from) them for a particular change in time are represented by linear functions. For these quantities, how much there is at any given point in time does not affect how much they will change by in the next time period.

In contrast, when a quantity changes in such a way that for a given change in time the quantity is always multiplied (or divided) by the same value, we have an exponential function. For these quantities, the more you have, the bigger the change in a given time period will be.

•Annual Percentage Change• In news reports, it is common to hear about the percent change of some quantity in comparison to a previous time. Often, these reports give the percent change since the previous year, as in these examples from the November 21, 2013 *New York Times* article "Homeless Tally Taken in January Found 13% Rise in New York."

(A) "In New York, where the shelter population has reached levels not seen since the Depression era, the count in January estimated 64,060 homeless people in shelters and on the street in January 2013, or 13 percent more than in January 2012."

(B) "Among large cities, only Los Angeles had a larger percentage increase. Its homeless population rose by 27 percent, although its total of 53,798 was lower than New York's."

(C) "Nationwide, the number of homeless people dropped by 4 percent from 2012, to 610,042 from 633,782."

In each of these examples, the percent given is a rise over the previous year, so it is an **annual percentage change**. In the New York and Los Angeles examples, we are not told how many homeless people there were in January 2012, but we can figure it out using the given information.

Example 3.2.1 Use the information in item (A) above to determine how many homeless people there were in New York City in January 2012.

Solution First, it is crucial to note that *the 13% in this example is 13% of the unknown January 2012 number, not 13% of 64,060*! Make sure you understand why this is the case and why it matters.

If N_o represents the number of homeless people in New York in January 2012, then there will be $0.13N_o$ *more* homeless people in January 2013. And we know that in January 2013 there were 64,060 homeless people, so we can write

$$N_o + 0.13N_o = 64,060 \tag{3.1}$$

To solve for N_o, we first factor it out of the expression on the left:

$$N_o(1 + 0.13) = 64,060 \tag{3.2}$$

Finally, we can divide both sides by 1.13 and round to the nearest person to obtain the result

$$N_o = 56,690 \tag{3.3}$$

It will be helpful in understanding various forms of exponential equations going forward to pay some attention now to the expression in parentheses in eq. (3.2). **The** 1 **serves the purpose of *keeping* the original number of homeless people and the** $+\ 0.13$ **serves the purpose of adding** 13% **of this amount.**

***Think About It* 3.2.1**
Use the information in item (B) above to determine how many homeless people there were in Los Angeles in January 2012.

***Think About It* 3.2.2**
Write an equation similar to eq. (3.2) which includes all of the information given in item (C) above. Check that the equation is true.

Exploration 3.8

Imagine the unlikely scenario where every year from January 2010 to January 2020 the annual percentage increase in New York City's homeless population was 13%. Determine New York's homeless population

E1. in January 2010

E2. in January 2020

E3. in July 2015

E4. t years after January 2010

Though annual percentage changes in various quantities are frequently reported in news stories, percentage increases over other time periods are certainly used as well. It can be useful, for the purposes of comparing percent increases given over different time periods, to determine the annual percentage change of each.

***Think About It* 3.2.3**
A Reuters article in the April 17, 2015 *New York Times* included the line "The so-called core C.P.I., which strips out food and energy costs, increased 0.2 percent in March after a similar rise in February." If this pattern were to continue for an entire year, what would the annual percentage change in the core C.P.I. be?

•Banking• In banking, the annual percentage change in your balance if you do not put any money into or take any money out of the account is known as the **APY**, which is short for **annual percentage yield**. If the bank compounds interest annually, this is the same value as what banks call the **APR**, which is short for **annual percentage rate**, but if the compounding is more often, the APY will be higher than the APR. If a bank compounds semiannually, instead of calculating your interest once a year at the APR, they award interest twice a year, determining how much to give you each time by multiplying your balance by half the APR. If a bank compounds quarterly, they award interest four times a year, determining how much to give you each time by multiplying your balance by one quarter of the APR.

***Think About It* 3.2.4**
Explain why the APY is higher than the APR if interest is compounded more than once a year.

Example 3.2.2 While today you'd be hard pressed to find a U.S. bank offering even a 2% APR, back in the early 1980s, APRs above 10% were common. (Of course, banks also charged much higher interest rates for those borrowing money in the 80s than they do today.) Find

(a) the APY if the APR is 12% and interest is compounded semiannually

(b) the amount you would have in the bank after 5 years of this interest scheme if you had deposited \$1000 initially

Solution

(a) While it's not absolutely necessary to introduce variables to answer this first question, we'll be glad to have them when we get to the next part, so let's call the initial balance A_o and let $A(t)$ be the amount in the account after t years. After half a year, we keep the initial amount, and add on 6% of it, so we have

$$A(0.5) = A_o(1 + 0.06)$$

and after another half year, we keep the amount we had after the first interest payment, $A(0.5)$, and add on 6% of this amount, so we have

$$\begin{aligned} A(1) &= A(0.5)(1 + 0.06) \\ &= A_o(1.06)(1.06) \\ &= A_o(1.06)^2 \\ &= 1.1236A_o \end{aligned}$$

Since the amount after 1 year is 1.1236 times the original amount, we know that our annual percentage change or APY is 12.36%. This means that a 12% APR compounded semiannually is equivalent to an annual compounding rate of 12.36%.

(b) If we think of this as getting 6% of our account balance twice a year for five years, we might write

$$\begin{aligned} A(5) &= 1000\left(1.06^2\right)^5 \\ &= 1000(1.06)^{10} \\ &= 1790.85 \end{aligned}$$

Alternatively, we can think of this of an annual percentage change of 12.36% for five years, in which case we might write

$$\begin{aligned} A(5) &= 1000(1.1236)^5 \\ &= 1790.85 \end{aligned}$$

Either way, we arrive at the conclusion that there will be a balance of \$1790.85 after five years.

***Think About It* 3.2.5**

Write an expression which can be evaluated to find $A(t)$ after t years if an amount A_o is deposited in an account that earns 12% interest compounded

(a) monthly (b) daily (c) n times per year

***Think About It* 3.2.6**

What do you think happens to the amount of money in your account as the value of n in item (c) of TAI 3.2.5 approaches infinity? That is, if you imagine compounding every minute or every second or every millisecond or every picosecond or …

***Think About It* 3.2.7**

What is the APY for an investment with an APR of 12% compounded continuously, as described in TAI 3.2.6? What is the APY for the general case in which the APR is r and interest is compounded continuously?

***Think About It* 3.2.8**

Explain the relationship between the graphs of $y = a(1.1274969)^x$ and $y = ae^{0.12x}$.

•The Continuous Growth Model• When we want to emphasize that a quantity is changing continuously rather than at discrete time intervals, we may choose to write our exponential equation describing the growth in the form

$$A(t) = A_o e^{rt} \tag{3.4}$$

and we refer the value of r in this case as the **continuous growth rate** or the **relative growth rate**. (In a banking context, we can, of course, continue to refer to it as an APR or annual rate and specify that the compounding is continuous.)

As we saw in TAI 3.2.8, we don't *have to* write our exponential equation in the form $A(t) = A_o e^{rt}$ just because we know the change is continuous. There is still an annual percentage change in the quantity A, and we can write an equation that produces the same graph in the form

$$A(t) = A_o(1+r)^t \tag{3.5}$$

It is essential to remain aware, however, that while A, t, and A_o represent the same thing in both forms of the equation, ***r* represents different things in eq. (3.4) and eq. (3.5)**. Assuming that time is measured in years, the r in eq. (3.4) is the continuous (or relative) annual growth rate of A, while the r in eq. (3.5) is the annual percentage change in A.

(In situations where the exponential model is not likely to be useful on a time scale of years, say in a laboratory experiment, a different unit of time may be chosen as the standard. If, for example, times used in the equation are in hours, the r in eq. (3.4) would be the continuous hourly growth rate, while the r in eq. (3.5) would be the hourly percentage change.)

***Think About It* 3.2.9**
Determine the continuous growth rate that will produce an annual percentage change of 25%. (Before doing any calculation, determine whether your answer should be higher or lower than 0.25.)

One situation which can be very well modeled by an exponential equation and where change is clearly continuous is that of radioactive decay. While scientists could have chosen to build up tables of values of annual (or hourly or …) percentage changes or continuous growth rates for the myriad radioactive elements, they have instead chosen to make tables of their **half-lives**. The half-life of a radioactive element is the time it takes for the mass of a sample of the element to be reduced by half.

Example 3.2.3 The element lawrencium-265 has a half life of 10 hours. Write an equation for $A(t)$, the mass of a sample of lawrencium-265 as a function of the time t (in hours) since it had a mass of A_o.

Solution One approach is to begin with eq. (3.4). While we don't know the continuous rate of decay, we do know that $A(10) = 0.5A_o$. (Why?)

$$\begin{aligned} A(10) &= A_o e^{r\cdot 10} \\ 0.5A_o &= A_o e^{10r} \\ 0.5 &= e^{10r} \\ \ln 0.5 &= 10r \\ r &= \frac{\ln 0.5}{10} \\ r &\approx -0.0693147 \end{aligned}$$

So one form of the equation for $A(t)$ is

$$A(t) = A_o e^{-0.0693147t} \tag{3.6}$$

Alternatively, we could begin with eq. (3.5), and again use the fact that $A(10) = 0.5A_o$:

$$\begin{aligned} A(10) &= A_o(1+r)^{10} \\ 0.5A_o &= A_o(1+r)^{10} \\ 0.5 &= (1+r)^{10} \\ 0.5^{1/10} &= 1+r \\ r &\approx -0.066967 \end{aligned}$$

So another form of the equation for $A(t)$ is

$$A(t) = A_o(1 - 0.06697)^t \tag{3.7}$$

But there is yet another approach to writing an equation when you're given the half-life, one which doesn't require any calculation at all. It requires recognizing that if time equal to two half-lives has passed, there will be $\frac{1}{4}$ of the original amount remaining, while if time equal to three half-lives has passed, there will be $\frac{1}{8}$ of the original amount remaining, and if time equal to n half-lives has passed, there will be $\left(\frac{1}{2}\right)^n$ of the original amount remaining. This yields the equation

$$A(t) = A_o\left(\frac{1}{2}\right)^{t/10} \tag{3.8}$$

Think About It **3.2.10**
Explain why the t in the exponent of eq. (3.8) is divided by the half-life.

Think About It **3.2.11**
Verify that eq. (3.6), eq. (3.7), and eq. (3.8) all give the same value for A when $t = 25$.

Think About It **3.2.12**
Write an equation a form similar to that of eq. (3.8) in order to model the population of a bacteria colony which has 1000 bacteria initially and triples in size every 14 hours.